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Mode III crack problems of two bonded functionally graded strips with internal cracks

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This paper deals with the anti-plane problem of two bonded functionally graded finite strips. Each strip contains an internal crack normal to the interface. The material properties of two strips are assumed to vary along the direction of the crack lines. A system of singular integral equations is derived and then solved numerically by using Gauss–Chebyshev integration formula. The influences of nonhomogeneous parameters, crack interactions and two edge conditions on the mode III stress intensity factors are investigated.

1. Introduction

To get rid of the abrupt change of material properties in laminate structures, functionally graded materials (FGMs) can be used to smooth the stress distribution. The crack problems of FGM layered elastic structures become attractive in the field of fracture mechanics.

The internal crack or cracks may be parallel or perpendicular to the interface. The studies of Noda and Jin (1993), Fotuhi and Fariborz (2006) and Wang et al. (2003) can be categorized to the former case. Several studies can be referred to the later case. Erdogan et al. (1991) solved the mode III crack problem in bonded two dissimilar homogeneous half planes with a nonhomogeneous interfacial zone. Choi (1996) studied the bonded dissimilar strips with a crack perpendicular to the functionally graded interface under the mode I loading. Erdogan and Wu (1997) solved the plane crack problem for a nonhomogeneous layer containing a crack. Ueda and Mukai (2002) studied the in-plane crack problem of a functionally graded nonhomogeneous interfacial layer. Surface layer with an internal crack is bonded to an interfacial layer and a substrate. Gao et al. (2004) studied the mode I crack problem in a functionally graded orthotropic strip. For the general case, Long and Delale (2004) presented a general problem for an arbitrarily oriented crack in a FGM layer. In these papers, the influence of the nonhomogeneous material properties, the geometry parameters on the stress intensity factors are discussed in detail. The studies of FGM layer structures have also extended to include the transient, viscoelastic and piezoelectric effects, such as Jin and Paulino (2002); Jin et al. (2003); Ueda (2005).

In this study, the stress field of two bonded functionally graded material strips is obtained. Each strip contains an internal crack perpendicular to the bonding surface. The material properties vary exponentially along crack line. Fourier transform is used to formulate the mode III crack problem into a system of singular integral equations, which is then solved by using shev Gauss–Chebyshev integration formula. Numerical results are graphically presented to illustrate the effects of

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the nonhomogeneous parameters, the crack interaction, strip geometry parameters, and the edge boundary conditions on the stress intensity factors.

2. Formulations

Fig. 1 shows the geometry of the structure. Two strips with width h_1 and h_2, respectively, are bonded along the interface $x = 0$. Each strip contains an internal crack with crack length $2a_{i0}$ perpendicular to the interface. The subscript i indicates the FGM strips 1 and 2. The shear loads applied at the crack surfaces are $\tau_1(x)$ and $\tau_2(x)$, respectively. The shear loads $\tau_1(x)$ and $\tau_2(x)$ can be obtained by using the principle of superposition from the external loads applied at infinity $y \rightarrow \pm\infty$. The shear moduli of both strips are assumed to vary along the x-axis. Under anti-plane deformation, the constitutive and equilibrium equations are as follows:

$$\tau_{jz(i)} = \mu_{(i)}(x) w_{(i),j} \tag{1}$$

$$\tau_{jz(i),j} = 0 \tag{2}$$

where $j = x, y$. The index i in the parentheses stand for the strips 1 and 2. The quantities $w_{(i)}$ and $\tau_{jz(i)}$ are the anti-plane displacements of strip i, and anti-plane shear stresses, respectively. The shear moduli $\mu_{(i)}(x)$ are assumed in the following exponential forms:

$$\mu_{(1)}(x) = \mu_0 \exp(\beta x) \quad (x > 0) \tag{3a}$$

$$\mu_{(2)}(x) = \mu_0 \exp(\gamma x) \quad (x < 0) \tag{3b}$$

where β and γ are the nonhomogeneous parameters of strips 1 and 2, respectively. The shear modulus μ_0 is assigned at the interface. Using Eqs. (3) and (1), the equilibrium Eq. (2) can be rewritten as

$$\left(\frac{\partial^2 w_{(1)}}{\partial x^2} + \frac{\partial^2 w_{(1)}}{\partial y^2}\right) + \beta \frac{\partial w_{(1)}}{\partial x} = 0 \tag{4a}$$

$$\left(\frac{\partial^2 w_{(2)}}{\partial x^2} + \frac{\partial^2 w_{(2)}}{\partial y^2}\right) + \gamma \frac{\partial w_{(2)}}{\partial x} = 0 \tag{4b}$$

Employing the Fourier transform on Eqs. (4a) and (4b), the solutions for w_1 and w_2 become

$$w_{(1)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{11}(\alpha, y) e^{-i\alpha x} d\alpha + \frac{2}{\pi} \int_0^{\infty} g_{11}(x, \alpha) \sin(\alpha y) d\alpha \tag{5a}$$

$$w_{(2)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{21}(\alpha, y) e^{-i\alpha x} d\alpha + \frac{2}{\pi} \int_0^{\infty} g_{21}(x, \alpha) \sin(\alpha y) d\alpha \tag{5b}$$

Since the problem is symmetric with respect to x-axis, only the upper half-plane $y > 0$ is considered here. From Eqs. (1), (2), (4) and (5) the unknown functions can be obtained as

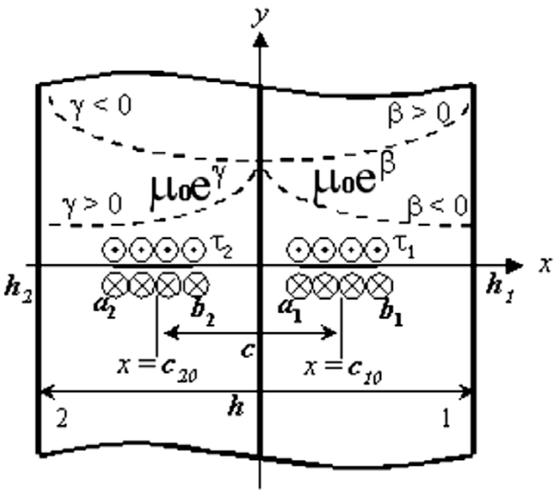

Fig. 1. Configuration of two bonded cracked FGM strips.

$$\begin{cases} f_{11}(\alpha, y) = A(\alpha)\exp(m_1 y) + A_2(\alpha)\exp(m_3 y) \\ f_{21}(\alpha, y) = B(\alpha)\exp(m_2 y) + B_2(\alpha)\exp(m_4 y) \\ g_{11}(x, \alpha) = C_1(\alpha)\exp(p_1 x) + C_2(\alpha)\exp(p_2 x) \\ g_{21}(x, \alpha) = D_1(\alpha)\exp(q_1 x) + D_2(\alpha)\exp(q_2 x) \end{cases} \tag{6a}$$

where

$$m_1 = -m_3 = -\sqrt{\alpha^2 + i\alpha\beta}$$
$$m_2 = -m_4 = -\sqrt{\alpha^2 + i\alpha\gamma}$$
$$p_1 = -\beta/2 - \alpha_1$$
$$p_2 = -\beta/2 + \alpha_1$$
$$q_1 = -\gamma/2 + \alpha_2$$
$$q_2 = -\gamma/2 - \alpha_2$$
$$\alpha_1 = \sqrt{\alpha^2 + \beta^2/4}$$
$$\alpha_2 = \sqrt{\alpha^2 + \gamma^2/4}$$

The roots $m_j (j = 1, 2 \cdots 4)$ are ordered in such a way that Re $(m_1) < 0$, *Re* $(m_2) < 0$ as α approach minus and plus infinity. From the regularity conditions at $y \to \infty$, the unknown functions in Eqs. (6a) should be rewritten as follows:

$$\begin{cases} f_{11}(\alpha, y) = A(\alpha)\exp(m_1 y) \\ f_{21}(\alpha, y) = B(\alpha)\exp(m_2 y) \\ g_{11}(x, \alpha) = C_1(\alpha)\exp(p_1 x) + C_2(\alpha)\exp(p_2 x) \\ g_{21}(x, \alpha) = D_1(\alpha)\exp(q_1 x) + D_2(\alpha)\exp(q_2 x) \end{cases} \tag{6b}$$

and $A(\alpha), \ldots, D_2(\alpha)$ are unknown functions to be obtained from the continuity and boundary conditions. The continuity conditions on the interface $x = 0$ are

$$w_{(1)}(0, y) = w_{(2)}(0, y) \tag{7a}$$
$$\tau_{xz(1)}(0, y) = \tau_{xz(2)}(0, y) \tag{7b}$$

The mixed boundary conditions on the $y = 0$ are as follows:

$$\tau_{yz(1)}(x, 0) = \tau_1(x) \quad \text{for } a_1 < x < b_1 \tag{8a}$$
$$\tau_{yz(2)}(x, 0) = \tau_2(x) \quad \text{for } a_2 < x < b_2 \tag{8b}$$

$$w_{(1)}(x, 0) = 0 \quad \text{for} \quad 0 \leqq x \leqq a_1 \text{ and } b_1 \leqq x < h_1 \tag{9a}$$
$$w_{(2)}(x, 0) = 0 \quad \text{for} \quad h_2 < x \leqq a_2 \text{and } b_2 \leqq x \leqq 0 \tag{9b}$$

The shear loads $\tau_1(x)$ and $\tau_2(x)$ are known functions and are obtained by using the principle of superposition from the external loads applied at infinity $y = \pm\infty$. Two dislocation functions are defined as follows (Erdogan, 1985):

$$g_1(x) = \frac{\partial}{\partial x} w_{(1)}(x, 0) \tag{10a}$$
$$g_2(x) = \frac{\partial}{\partial x} w_{(2)}(x, 0) \tag{10b}$$

which must satisfy the following single-valued conditions:

$$\int_{a_1}^{b_1} g_1(t)\mathrm{d}t = \int_{a_2}^{b_2} g_2(t)\mathrm{d}t = 0 \tag{11}$$

The unknown functions $A(\alpha)$ and $B(\alpha)$ in Eq. (6b) can be expressed in the form of dislocation functions as

$$A(\alpha) = \frac{i}{\alpha}\int_{a_1}^{b_1} g_1(t)e^{i\alpha t}\mathrm{d}t \tag{12a}$$
$$B(\alpha) = \frac{i}{\alpha}\int_{a_2}^{b_2} g_2(t)e^{i\alpha t}\mathrm{d}t \tag{12b}$$

After employing the continuity conditions Eq. (7) and the Fourier inverse transform, we obtain the following relations among four unknown functions $C_1(\alpha), \ldots, D_2(\alpha)$:

$$D_1(\alpha) + D_2(\alpha) - C_1(\alpha) - C_2(\alpha) = R_1(\alpha) \tag{13a}$$

$$q_1 D_1(\alpha) + q_2 D_2(\alpha) - p_1 C_1(\alpha) - p_2 C_2(\alpha) = R_2(\alpha) \tag{13b}$$

with

$$R_1(\alpha) = \frac{\alpha}{2\alpha_1 p_2} \int_{a_1}^{b_1} g_1(t) e^{-p_2 t} dt - \frac{\alpha}{2\alpha_2 q_2} \int_{a_2}^{b_2} g_2(t) e^{-q_2 t} dt \tag{14a}$$

$$R_2(\alpha) = \frac{\alpha}{2\alpha_1} \int_{a_1}^{b_1} g_1(t) e^{-p_2 t} dt - \frac{\alpha}{2\alpha_2} \int_{a_2}^{b_2} g_2(t) e^{-q_2 t} dt \tag{14b}$$

Depending on the boundary conditions at the surfaces $x = h_1$ and $x = h_2$, two cases are discussed in this study to get the functions C_1, C_2, D_1, and D_2 in terms of the dislocation functions g_1 and g_2.

Case 1: Two edge surfaces are free of traction

The two edge surfaces at $x = h_1$ and $x = h_2$ are assumed to be free of traction as

$$\tau_{xz(1)}(h_1, y) = \tau_{xz(2)}(h_2, y) = 0 \tag{15}$$

From the conditions (15), the following relations can be obtained after performing the Fourier inverse transform:

$$C_1(\alpha) p_1 e^{p_1 h_1} + C_2(\alpha) p_2 e^{p_2 h_1} = -\frac{\alpha}{2\alpha_1} \int_{a_1}^{b_1} g_1(t) e^{p_1(h_1 - t)} dt \tag{16a}$$

$$D_1(\alpha) q_1 e^{q_1 h_2} + D_2(\alpha) q_2 e^{q_2 h_2} = -\frac{\alpha}{2\alpha_2} \int_{a_2}^{b_2} g_2(t) e^{q_2(h_2 - t)} dt \tag{16b}$$

By solving Eqs. (13) and (16) for functions C_1, C_2, D_1, and D_2, we have

$$\begin{aligned} C_1(\alpha) = \ & \tfrac{1}{\Delta}\{\alpha\alpha_2 e^{p_2 h_1}[q_1 q_2(e^{q_2 h_2} - e^{q_1 h_2}) + p_2(q_1 e^{q_1 h_2} - q_2 e^{q_2 h_2})] \textstyle\int_{a_1}^{b_1} g_1(t) e^{-p_2 t} dt \\ & -\alpha\alpha_2 e^{p_1 h_1}[p_2(q_1 e^{q_1 h_2} - q_2 e^{q_2 h_2}) + q_1 q_2(e^{q_2 h_2} - e^{q_1 h_2})] \textstyle\int_{a_1}^{b_1} g_1(t) e^{-p_1 t} dt \\ & +\alpha\alpha_1 p_2 e^{p_2 h_1} e^{q_2 h_2}(q_2 - q_1) \textstyle\int_{a_2}^{b_2} g_2(t) e^{-q_2 t} dt\} + \alpha\alpha_1 p_2 e^{p_2 h_1} e^{q_1 h_2}(q_1 - q_2) \textstyle\int_{a_2}^{b_2} g_2(t) e^{-q_1 t} dt\} \end{aligned} \tag{17a}$$

$$\begin{aligned} C_2(\alpha) = \ & \tfrac{1}{p_2 \Delta}\{\alpha\alpha_2 e^{p_1 h_1}[p_1 p_2(q_2 e^{q_2 h_2} - q_1 e^{q_1 h_2}) + q_1 q_2 p_1(e^{q_1 h_2} - e^{q_2 h_2})] \textstyle\int_{a_1}^{b_1} g_1(t) e^{-p_2 t} dt \\ & +\alpha\alpha_2 e^{p_1 h_1}[p_1 p_2(q_1 e^{q_1 h_2} - q_2 e^{q_2 h_2}) + q_1 q_2 p_2(e^{q_2 h_2} - e^{q_1 h_2})] \textstyle\int_{a_1}^{b_1} g_1(t) e^{-p_1 t} dt \\ & +\alpha\alpha_1 p_1 p_2 e^{p_1 h_1} e^{q_2 h_2}(q_1 - q_2) \textstyle\int_{a_2}^{b_2} g_2(t) e^{-q_2 t} dt + \alpha\alpha_1 p_1 p_2 e^{p_1 h_1} e^{q_1 h_2}(q_2 - q_1) \textstyle\int_{a_2}^{b_2} g_2(t) e^{-q_1 t} dt\} \end{aligned} \tag{17b}$$

$$\begin{aligned} D_1(\alpha) = \ & \tfrac{1}{\Delta}\{\alpha\alpha_2 q_2 e^{q_2 h_2} e^{p_2 h_1}(p_1 - p_2) \textstyle\int_{a_1}^{b_1} g_1(t) e^{-p_2 t} dt - \alpha\alpha_2 q_2 e^{q_2 h_2} e^{p_1 h_1}(p_1 - p_2) \textstyle\int_{a_1}^{b_1} g_1(t) e^{-p_1 t} dt \\ & +\alpha\alpha_1 e^{q_2 h_2}[p_1 p_2(e^{p_1 h_1} - e^{p_2 h_1}) + q_2(p_2 e^{p_2 h_1} - p_1 e^{p_1 h_1})] \textstyle\int_{a_2}^{b_2} g_2(t) e^{-q_2 t} dt \\ & +\alpha\alpha_1 e^{q_1 h_2}[p_1 p_2(e^{p_2 h_1} - e^{p_1 h_1}) + q_2(p_1 e^{p_1 h_1} - p_2 e^{p_2 h_1})] \textstyle\int_{a_2}^{b_2} g_2(t) e^{-q_1 t} dt\} \end{aligned} \tag{17c}$$

$$\begin{aligned} D_2(\alpha) = \ & \tfrac{1}{q_2 \Delta}\{\alpha\alpha_2 q_1 q_2 e^{q_1 h_2} e^{p_2 h_1}(p_2 - p_1) \textstyle\int_{a_1}^{b_1} g_1(t) e^{-p_2 t} dt + \alpha\alpha_2 q_1 q_2 e^{q_1 h_2} e^{p_1 h_1}(p_1 - p_2) \textstyle\int_{a_1}^{b_1} g_1(t) e^{-p_1 t} dt \\ & +\alpha\alpha_1 q_1 e^{q_1 h_2}[p_1 p_2(e^{p_2 h_1} - e^{p_1 h_1}) + q_2(p_1 e^{p_1 h_1} - p_2 e^{p_2 h_1})] \textstyle\int_{a_2}^{b_2} g_2(t) e^{-q_2 t} dt \\ & +\alpha\alpha_1 q_2 e^{q_1 h_2}[p_1 p_2(e^{p_1 h_1} - e^{p_2 h_1}) + q_1(p_2 e^{p_2 h_1} - p_1 e^{p_1 h_1})] \textstyle\int_{a_2}^{b_2} g_2(t) e^{-q_1 t} dt\} \end{aligned} \tag{17d}$$

where

$$\Delta = -2\alpha_1\alpha_2\alpha^2\left[(e^{p_1 h_1} - e^{p_2 h_1})(q_1 e^{q_1 h_2} - q_2 e^{q_2 h_2}) + (e^{q_2 h_2} - e^{q_1 h_2})(p_1 e^{p_1 h_1} - p_2 e^{p_2 h_1})\right]$$

From the load conditions (8) on the crack surfaces, the following equations can be obtained:

$$\tau_{yz(1)}(x, 0) = \tau_1(x) = \mu_0 e^{\beta x}\left\{\begin{array}{l} \frac{1}{\pi}\int_{a_1}^{b_1}[k_1(x,t) + k_2(x,t) + k_3(x,t) + k_4(x,t) + k_5(x,t)]g_1(t)dt \\ +\frac{1}{\pi}\int_{a_2}^{b_2}[k_6(x,t) + k_7(x,t) + k_8(x,t) + k_9(x,t)]g_2(t)dt \end{array}\right\} \tag{18a}$$

$$\tau_{yz(2)}(x, 0) = \tau_2(x) = \mu_0 e^{\gamma x}\left\{\begin{array}{l} \frac{1}{\pi}\int_{a_2}^{b_2}[k_{10}(x,t) + k_{11}(x,t) + k_{12}(x,t) + k_{13}(x,t) + k_{14}(x,t)]g_2(t)dt \\ +\frac{1}{\pi}\int_{a_1}^{b_1}[k_{15}(x,t) + k_{16}(x,t) + k_{17}(x,t) + k_{18}(x,t)]g_1(t)dt \end{array}\right\} \tag{18b}$$

where the kernels k_i (x, t), ($i = 1, \ldots, 18$) are given in Appendix A. Separating the singular term of the kernels $k_1(x, t)$ and $k_{10}(x, t)$, Eq. (18) may be rewritten as follows:

$$\tau_{yz(1)}(x,0)=\mu_0 e^{\beta x}\left\{\begin{array}{l}\frac{1}{\pi}\int_{a_1}^{b_1}\left[\frac{1}{t-x}+h_1(x,t)+k_2(x,t)+k_3(x,t)+k_4(x,t)+k_5(x,t)\right]g_1(t)\mathrm{d}t\\ +\frac{1}{\pi}\int_{a_2}^{b_2}[k_6(x,t)+k_7(x,t)+k_8(x,t)+k_9(x,t)]g_2(t)\mathrm{d}t\end{array}\right\} \tag{19a}$$

$$\tau_{yz(2)}(x,0)=\mu_0 e^{\gamma x}\left\{\begin{array}{l}\frac{1}{\pi}\int_{a_2}^{b_2}\left[\frac{1}{t-x}+h_{10}(x,t)+k_{11}(x,t)+k_{12}(x,t)+k_{13}(x,t)+k_{14}(x,t)\right]g_2(t)\mathrm{d}t\\ +\frac{1}{\pi}\int_{a_1}^{b_1}[k_{15}(x,t)+k_{16}(x,t)+k_{17}(x,t)+k_{18}(x,t)]g_1(t)\mathrm{d}t\end{array}\right\} \tag{19b}$$

where

$$h_1(x,t)=\mathrm{Im}\left\{\int_0^\infty\left(\sqrt{1+\frac{i\beta}{\alpha}}-1\right)e^{i\alpha(t-x)}\mathrm{d}\alpha\right\}$$

$$h_{10}(x,t)=\mathrm{Im}\left\{\int_0^\infty\left(\sqrt{1+\frac{i\gamma}{\alpha}}-1\right)e^{i\alpha(t-x)}\mathrm{d}\alpha\right\}$$

These two equations are called the singular integral equations of the first kind with simple Cauchy-type singularities. All kernels in Eq. (19) are bounded except the term $1/(t-x)$, which contribute the singular effects. The dislocation functions g_1 and g_2 can be solved numerically by using Gauss–Chebyshev integration formula.

Case 2: Two edge surfaces are fixed

The two surfaces at $x = h_1$ and $x = h_2$ are assumed to be fixed as follows:

$$w_{(1)}(h_1,y)=w_{(2)}(h_2,y)=0 \tag{20}$$

Following the deriving procedures of Case 1, the following relations can be obtained:

$$C_1(\alpha)e^{p_1h_1}+C_2(\alpha)e^{p_2h_1}=-\frac{\alpha}{2\alpha_1p_1}\int_{a_1}^{b_1}g_1(t)e^{p_1(h_1-t)}\mathrm{d}t \tag{21a}$$

$$D_1(\alpha)e^{q_1h_2}+D_2(\alpha)e^{q_2h_2}=-\frac{\alpha}{2\alpha_2q_1}\int_{a_2}^{b_2}g_2(t)e^{q_2(h_2-t)}\mathrm{d}t \tag{21b}$$

By solving Eqs. (13) and (21) for functions C_1, C_2, D_1, and D_2, we have:

$$\begin{aligned}C_1(\alpha)=\;&\tfrac{1}{\Delta_1}\{\alpha\alpha_2p_1q_1q_2e^{p_2h_1}[p_2(e^{q_2h_2}-e^{q_1h_2})+(q_2e^{q_1h_2}-q_1e^{q_2h_2})]\textstyle\int_{a_1}^{b_1}g_1(t)e^{-p_2t}\mathrm{d}t\\ &-\alpha\alpha_2p_2q_1q_2e^{p_1h_1}[p_2(e^{q_2h_2}-e^{q_1h_2})+(q_2e^{q_1h_2}-q_1e^{q_2h_2})]\textstyle\int_{a_1}^{b_1}g_1(t)e^{-p_1t}\mathrm{d}t\\ &+\alpha\alpha_1p_1p_2q_1e^{p_2h_1}e^{q_2h_2}(q_1-q_2)\textstyle\int_{a_2}^{b_2}g_2(t)e^{-q_2t}\mathrm{d}t\}+\alpha\alpha_1p_1p_2q_2e^{p_2h_1}e^{q_1h_2}(q_2-q_1)\textstyle\int_{a_2}^{b_2}g_2(t)e^{-q_1t}\mathrm{d}t\}\end{aligned} \tag{22a}$$

$$\begin{aligned}C_2(\alpha)=\;&\tfrac{1}{\Delta_1}\{\alpha\alpha_2p_1q_1q_2e^{p_1h_1}[p_2(e^{q_1h_2}-e^{q_2h_2})+(q_1e^{q_2h_2}-q_2e^{q_1h_2})]\textstyle\int_{a_1}^{b_1}g_1(t)e^{-p_2t}\mathrm{d}t\\ &+\alpha\alpha_2p_2q_1q_2e^{p_1h_1}[p_1(e^{q_2h_2}-e^{q_1h_2})+(q_2e^{q_1h_2}-q_1e^{q_2h_2})]\textstyle\int_{a_1}^{b_1}g_1(t)e^{-p_1t}\mathrm{d}t\\ &+\alpha\alpha_1p_1p_2q_1e^{p_1h_1}e^{q_2h_2}(q_2-q_1)\textstyle\int_{a_2}^{b_2}g_2(t)e^{-q_2t}\mathrm{d}t+\alpha\alpha_1p_1p_2q_2e^{p_1h_1}e^{q_1h_2}(q_1-q_2)\textstyle\int_{a_2}^{b_2}g_2(t)e^{-q_1t}\mathrm{d}t\}\end{aligned} \tag{22b}$$

$$\begin{aligned}D_1(\alpha)=\;&\tfrac{1}{\Delta_1}\{\alpha\alpha_2p_1q_1q_2e^{q_2h_2}e^{p_2h_1}(p_2-p_1)\textstyle\int_{a_1}^{b_1}g_1(t)e^{-p_2t}\mathrm{d}t-\alpha\alpha_2p_2q_1q_2e^{q_2h_2}e^{p_1h_1}(p_2-p_1)\textstyle\int_{a_1}^{b_1}g_1(t)e^{-p_1t}\mathrm{d}t\\ &+\alpha\alpha_1p_1p_2q_1e^{q_2h_2}[q_2(e^{p_1h_1}-e^{p_2h_1})+(p_1e^{p_2h_1}-p_2e^{p_1h_1})]\textstyle\int_{a_2}^{b_2}g_2(t)e^{-q_2t}\mathrm{d}t\\ &+\alpha\alpha_1p_1p_2q_2e^{q_1h_2}[q_2(e^{p_2h_1}-e^{p_1h_1})+(p_2e^{p_1h_1}-p_1e^{p_2h_1})]\textstyle\int_{a_2}^{b_2}g_2(t)e^{-q_1t}\mathrm{d}t\}\end{aligned} \tag{22c}$$

$$\begin{aligned}D_2(\alpha)=\;&\tfrac{1}{\Delta_1}\{\alpha\alpha_2p_1q_1q_2e^{q_1h_2}e^{p_2h_1}(p_1-p_2)\textstyle\int_{a_1}^{b_1}g_1(t)e^{-p_2t}\mathrm{d}t+\alpha\alpha_2p_2q_1q_2e^{q_1h_2}e^{p_1h_1}(p_2-p_1)\textstyle\int_{a_1}^{b_1}g_1(t)e^{-p_1t}\mathrm{d}t\\ &+\alpha\alpha_1p_1p_2q_1e^{q_1h_2}[q_2(e^{p_2h_1}-e^{p_1h_1})+(p_2e^{p_1h_1}-p_1e^{p_2h_1})]\textstyle\int_{a_2}^{b_2}g_2(t)e^{-q_2t}\mathrm{d}t\\ &+\alpha\alpha_1p_1p_2q_2e^{q_1h_2}[q_1(e^{p_1h_1}-e^{p_2h_1})+(p_1e^{p_2h_1}-p_2e^{p_1h_1})]\textstyle\int_{a_2}^{b_2}g_2(t)e^{-q_1t}\mathrm{d}t\}\end{aligned} \tag{22d}$$

where

$$\Delta_1=2p_1p_2q_1q_2\alpha_1\alpha_2\left[(e^{q_1h_2}-e^{q_2h_2})(p_1e^{p_2h_1}-p_2e^{p_1h_1})+(e^{p_1h_1}-e^{p_2h_1})(q_2e^{q_1h_2}-q_1e^{q_2h_2})\right]$$

From the load conditions (8) on the crack surfaces, the governing equations for dislocation functions g_1 and g_2 are the same as Eq. (19) of Case 1 with different kernels given in Appendix B.

3. Degenerated problems

The crack problem governed by Eq. (19) can be degenerated to some simple problems by changing the geometric parameters. Three degenerated problems reduced from Case 1 are discussed.

3.1. Degenerated problem 1

A FGM strip bounded to a FGM medium ($-h_2 \to \infty$)

If the edge boundary with $x = h_2$ moves to minus infinity ($-h_2 \to \infty$), the kernels $k_8(x, t)$, $k_9(x, t)$, $k_{12}(x, t)$, $k_{13}(x, t)$, $k_{14}(x, t)$, $k_{17}(x, t)$ and $k_{18}(x, t)$ in Eq. (19) go to zero, and the results become:

$$\tau_{yz(1)}(x,0) = \mu_0 e^{\beta x}\left\{\begin{array}{l} \frac{1}{\pi}\int_{a_1}^{b_1}[k_1(x,t)+k_2(x,t)+k_3(x,t)+k_4(x,t)+k_5(x,t)]g_1(t)dt \\ +\frac{1}{\pi}\int_{a_2}^{b_2}[k_6(x,t)+k_7(x,t)]g_2(t)dt \end{array}\right\} \tag{23a}$$

$$\tau_{yz(2)}(x,0) = \mu_0 e^{\gamma x}\left\{\frac{1}{\pi}\int_{a_2}^{b_2}[k_{10}(x,t)+k_{11}(x,t)]g_2(t)dt + \frac{1}{\pi}\int_{a_1}^{b_1}[k_{15}(x,t)+k_{16}(x,t)]g_1(t)dt\right\} \tag{23b}$$

where the remaining kernels k_i (x,t), (i = 1–7, 10, 11, 15, 16) should be modified and are shown in Appendix C.

3.2. Degenerated problem 2

Two bonded cracked FGM half planes ($h_1 \to \infty$, $-h_2 \to \infty$). If the edge boundaries h_1 and h_2 move to infinity ($h_1 \to \infty$, $-h_2 \to \infty$), the kernels $k_3(x, t)$, $k_4(x, t)$, $k_5(x, t)$, $k_7(x, t)$, $k_8(x, t)$, $k_9(x, t)$, $k_{12}(x, t)$, $k_{13}(x, t)$, $k_{14}(x, t)$, $k_{16}(x, t)$, $k_{17}(x, t)$ and $k_{18}(x, t)$ in Eq. (19) will disappear, and the results become:

$$\tau_{yz(1)}(x,0) = \mu_0 e^{\beta x}\left\{\frac{1}{\pi}\int_{a_1}^{b_1}\left[\frac{1}{t-x}+h_1(x,t)+k_2(x,t)\right]g_1(t)dt + \frac{1}{\pi}\int_{a_2}^{b_2}[k_6(x,t)]g_2(t)dt\right\} \tag{24a}$$

$$\tau_{yz(2)}(x,0) = \mu_0 e^{\gamma x}\left\{\frac{1}{\pi}\int_{a_2}^{b_2}\left[\frac{1}{t-x}+h_{10}(x,t)+k_{11}(x,t)\right]g_2(t)dt + \frac{1}{\pi}\int_{a_1}^{b_1}[k_{15}(x,t)]g_1(t)dt\right\} \tag{24b}$$

The kernels k_i (x, t), (i = 2, 6, 11, 15) in the above equations are as follows:

$$k_2(x,t) = e^{\frac{\beta}{2}(t-x)}\int_0^\infty \frac{\alpha^2(q_1-p_2)}{p_2\alpha_1(p_1-q_1)}e^{-\alpha_1(t+x)}d\alpha$$

$$k_6(x,t) = e^{(\frac{\gamma}{2}t-\frac{\beta}{2}x)}\int_0^\infty \frac{\alpha^2(q_2-q_1)}{q_2\alpha_2(p_1-q_1)}e^{(\alpha_2 t-\alpha_1 x)}d\alpha$$

$$k_{11}(x,t) = e^{\frac{\gamma}{2}(t-x)}\int_0^\infty \frac{\alpha^2(q_2-p_1)}{\alpha_2 q_2(p_1-q_1)}e^{\alpha_2(t+x)}d\alpha$$

$$k_{15}(x,t) = e^{(\frac{\beta}{2}t-\frac{\gamma}{2}x)}\int_0^\infty \frac{\alpha^2(p_1-p_2)}{\alpha_1 p_2(p_1-q_1)}e^{(-\alpha_1 t+\alpha_2 x)}d\alpha$$

It can be seen that the existence of kernels $k_3(x, t)$, $k_4(x, t)$, $k_5(x, t)$, $k_7(x, t)$, $k_8(x, t)$, $k_9(x, t)$, $k_{12}(x, t)$, $k_{13}(x, t)$, $k_{14}(x, t)$, $k_{16}(x, t)$, $k_{17}(x, t)$ and $k_{18}(x, t)$ in Eq. (19) come from the effects of the boundary surfaces ($x = h_1$, h_2) acted on the two cracks.

3.3. Degenerated problem 3

($h_1 \to \infty$, $-h_2 \to \infty$, $2a_{20} = 0$)

Consider the case that the boundary surfaces move to infinity and the crack in material 2 vanishes. It becomes a cracked FGM half plane bonded to a FGM half plane. The governing equation Eq. (24a) for $g_1$is reduced to the following form:

$$\tau_{yz(1)}(x,0) = \mu_0 e^{\beta x}\frac{1}{\pi}\int_{a_1}^{b_1}\left[\frac{1}{t-x}+h_1(x,t)+k_2(x,t)\right]g_1(t)dt \tag{24c}$$

It agrees with Eq. (20) of Erdogan (1985).

4. Solutions of the singular integral equations

The solutions of the singular integral equations Eq. (19) with the Cauchy type kernel are:

$$g_i(t) = \frac{G_i(t)}{\sqrt{(t-a_i)(b_i-t)}} \quad (i=1,2) \tag{25}$$

where $G_j(t)$ are bounded functions. The stress intensity factors at the crack tips are obtained as

$$k_3(b_1) = \lim_{x\to b_1^+}\sqrt{2(x-b_1)}\tau_{yz(1)}(x,0) = -\mu_0 e^{\beta b_1}\frac{G_1(b_1)}{\sqrt{(b_1-a_1)/2}} \tag{26a}$$

$$k_3(a_1) = \lim_{x \to a_1^-} \sqrt{2(a_1 - x)}\tau_{yz(1)}(x, 0) = \mu_0 e^{\beta a_1} \frac{G_1(a_1)}{\sqrt{(b_1 - a_1)/2}} \tag{26b}$$

$$k_3(b_2) = \lim_{x \to b_2^+} \sqrt{2(x - b_2)}\tau_{yz(2)}(x, 0) = -\mu_0 e^{\gamma b_2} \frac{G_2(b_2)}{\sqrt{(b_2 - a_2)/2}} \tag{26c}$$

$$k_3(a_2) = \lim_{x \to a_2^-} \sqrt{2(a_2 - x)}\tau_{yz(2)}(x, 0) = \mu_0 e^{\gamma a_2} \frac{G_2(a_2)}{\sqrt{(b_2 - a_2)/2}} \tag{26d}$$

In deriving the above equations, the following relations have been used (Muskhhelishvili, 1953):

$$\frac{1}{\pi}\int_{a_1}^{b_1} \frac{g_1(t)}{t - x} dt = \frac{G_1(a_1)e^{\frac{1}{2}\pi i}}{\sqrt{b_1 - a_1}\sqrt{x - a_1}} - \frac{G_1(b_1)}{\sqrt{b_1 - a_1}\sqrt{x - b_1}} + \text{Other terms} \tag{27a}$$

$$\frac{1}{\pi}\int_{a_2}^{b_2} \frac{g_2(t)}{t - x} dt = \frac{G_2(a_2)e^{\frac{1}{2}\pi i}}{\sqrt{b_2 - a_2}\sqrt{x - a_2}} - \frac{G_2(b_2)}{\sqrt{b_2 - a_2}\sqrt{x - b_2}} + \text{Other terms} \tag{27b}$$

In order to obtain the specific functions G_1 (a_1), $G_1(b_1)$, $G_2(a_2)$ and $G_2(b_2)$, we define dimensionless quantities (Erdogan et al., 1973) as follows:

$$\bar{x}_i = \frac{x_i - c_{i0}}{a_{i0}}, \quad \bar{t}_i = \frac{t_i - c_{i0}}{a_{i0}}, \quad \bar{h}_i = \frac{h_i - c_{i0}}{a_{i0}}, \quad f_i(\bar{t}_i) = g_i(t), \quad (i = 1, 2)$$

then Eq. (25) become:

$$f_i(\bar{t}_i) = \frac{F_i(\bar{t}_i)}{\sqrt{(1 + \bar{t}_i)(1 - \bar{t}_i)}}, \quad (i = 1, 2) \tag{28}$$

where $F_1(\overline{t_1})$ and $F_2(\overline{t_2})$ are related to $G_1(t)$ and $G_2(t)$. Eq. (19) can be expressed in Chebyshev polynomials, which are expressed in Appendix D. The results of stress intensity factors become:

$$k_3(b_i) = -\mu_0 e^{\beta b_i}\sqrt{a_{i0}}F_i(1) \tag{29a}$$

$$k_3(a_i) = \mu_0 e^{\beta a_i}\sqrt{a_{i0}}F_i(-1) \tag{29b}$$

with i = 1, 2. Based on the quadratic extrapolation technique, the unknown crack tip values of $F_i(1)$ and $F_i(-1)$ can be obtained by using the values of F_i at nodes 2, 3, 4 and $n-1$, $n-2$, $n-3$, respectively. Here n is the number of collocation points along crack lines.

5. Results and discussions

In the following numerical computations, the shear loads τ_1 and τ_2 applied on the crack surfaces are assumed to be equal. Two crack lengths $2a_{20}$ and $2a_{10}$ are also assumed to be equal. The stress intensity factors at the crack tips are normalized as:

$$k_{ai} = \frac{k_3(a_i)}{\tau_2\sqrt{a_{20}}} \tag{30a}$$

$$k_{bi} = \frac{k_3(b_i)}{\tau_2\sqrt{a_{20}}} \tag{30b}$$

with i = 1, 2.

5.1. A FGM thin film bonded to a homogeneous elastic substrate ($-h_2 \to \infty$, $\gamma = 0$)

Consider a practical case that a FGM thin film is bonded to a homogeneous elastic substrate. The material properties of this thin film can be selected to fit the functional requirement. The left crack is located at $-c_{20}/h_1 = 1/5$. Fig. 2(a) and (b) show the variations of normalized intensity factors with normalized length c_{10}/h_1 at different values of βa_{10} when the boundary surface $x = h_1$ is free of traction.

In Fig. 2(a), k_{b1} is greater than k_{a1} when βa_{10} is positive and vice versa. Because of the crack interaction, k_{a1} increases as the crack approaches the interface. Consider the case $\beta a_{10} = 0$. Due to the edge effect, k_{b1} is greater than k_{a1} when the crack is close to the film surface. As the right crack moves to the interface $x = 0$, k_{a1} becomes larger and finally k_{a1} is greater than k_{b1} according to the crack interaction effect.

For all values of βa_{10} in Fig. 2(b), a general tendency can be seen that both factors k_{b2} and k_{a2} decrease and approach to a constant as c_{10}/h_1 increases. It is due to the gradually decay of the crack interaction effect.

5.2. Two bonded FGM cracked-strips

Let us go back to the most general case that two cracked strips are bonded together. In the following discussion, the geometry parameters of the left crack are kept unchanged. The ratios of c_{10}/h and $h_1/-h_2$ are subjected to change in order to study the effects of crack location and edge conditions, respectively.

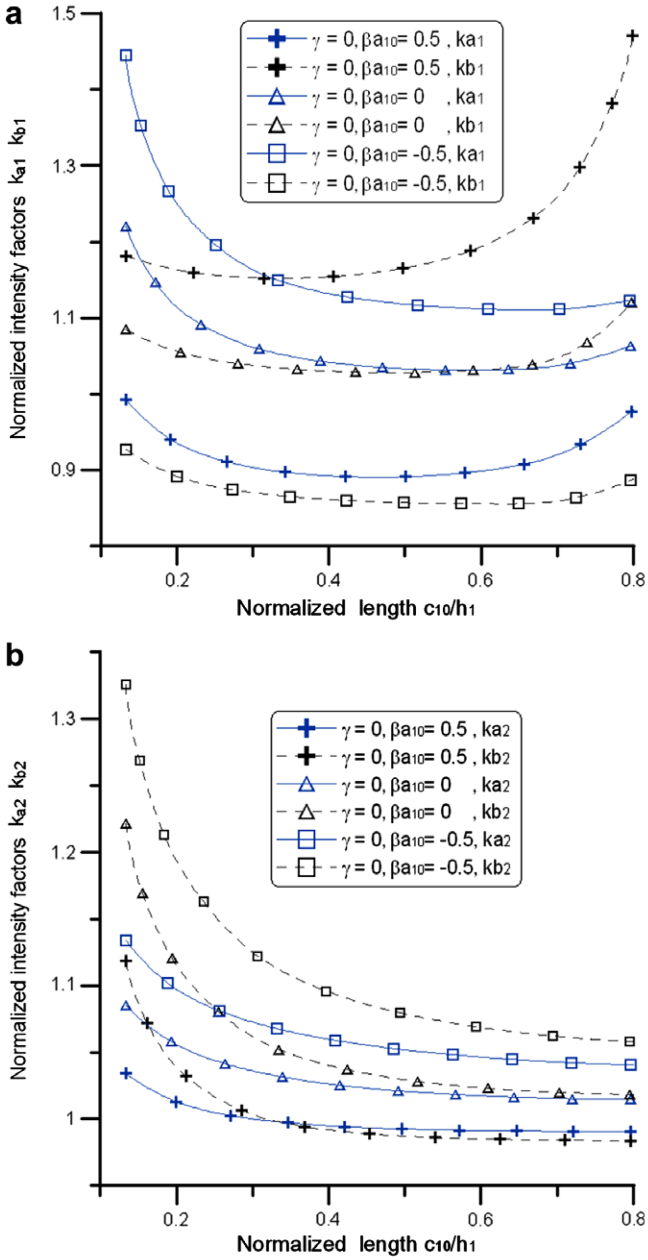

Fig. 2. Variations of normalized intensity factors with c_{10}/h_1 when $\gamma = 0$, $h_2 \to -\infty$.

5.2.1. Effects of crack location

Fig. 3(a) and (b) are the variations of normalized intensity factors with c_{10}/h when $\beta a_{10} = -\gamma a_{20} = -0.25$, and $\beta a_{10} = -\gamma a_{20} = 0.25$, respectively. The thickness of the strips $h_1 = -h_2$. The geometry parameters of the left crack are $h_2/c_{20} = 5$ and $h_2/(-2a_{20}) = 3.75$. From the FGM definition Eq. (3), the stiffness at the interface is highest when $\beta a_{10} = -\gamma a_{20} = -0.25$, while it is weakest for $\beta a_{10} = -\gamma a_{20} = 0.25$.

In Fig. 3(a), k_{a1} and k_{b2} are greater than k_{b1} and k_{a2}because of the crack interaction and material effects. As the right crack moves to the right with increasing c_{10}/h, k_{a2} and k_{b2} decrease and reach to a constant value, respectively.

Consider the case in Fig. 3(b). Since the material effect dominates, k_{a2} is always greater than k_{b2}. The difference between them is reduced smaller when crack distance c_{10}/h is small and crack interaction becomes prominent. Same conclusion can be made to the factors k_{a1} and k_{b1}. However, due to the edge effect, these two factors increase for Case 1 and decrease for Case 2 when the right crack approaches the boundary $x = h_1$.

5.2.2. Effects of edge boundary conditions

Let us consider the case that the configuration of Fig. 1 remain unchanged except the right boundary surface $x = h_1$. This surface is extended to the right. Under the conditions $h_2/(-2a_{20}) = 3.75$ and $h_2/c_{20} = -h_2/c_{10} = 2$, the variations of normalized stress intensity factor k_{b1} shown in Fig. 4 are examined to study the edge effects come from edge conditions and the normalized length parameter $h_1/-h_2$.

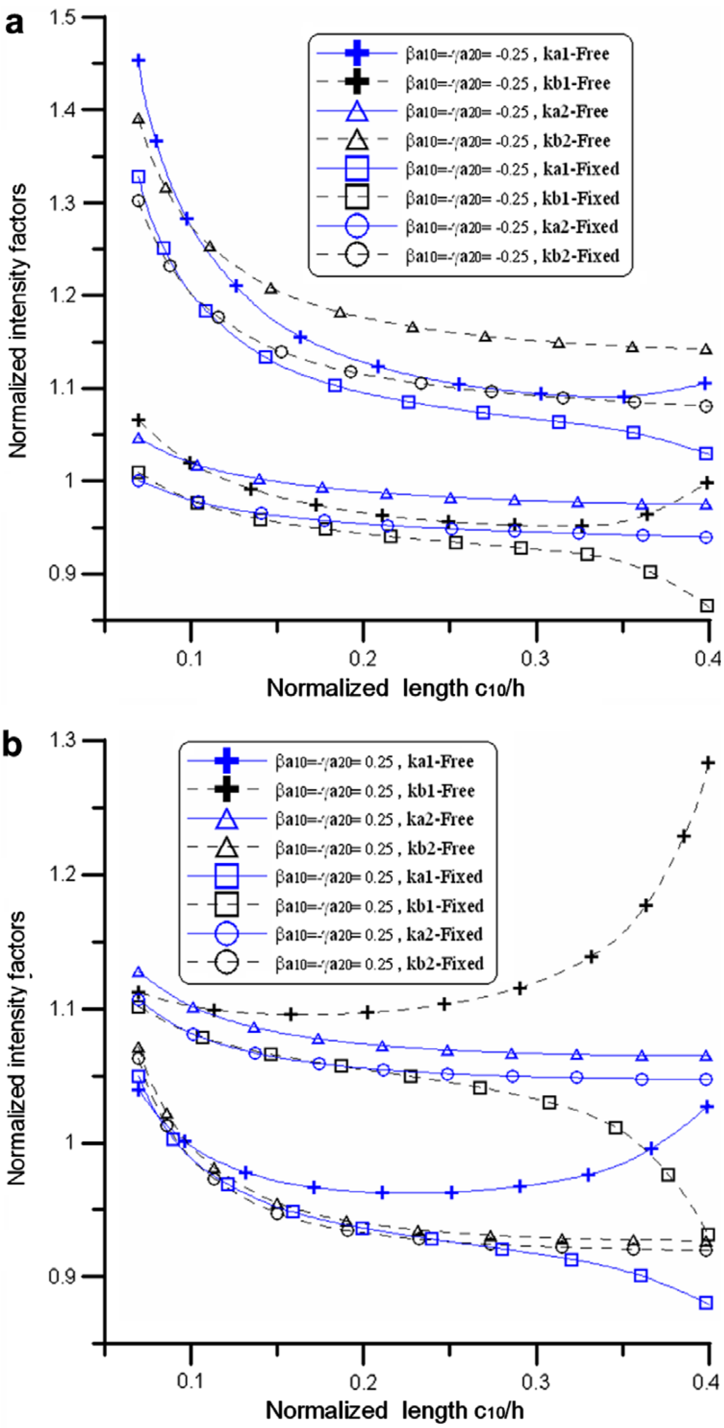

Fig. 3. Variations of normalized intensity factors with c_{10}/h when (a) $\beta a_{10} = -\gamma a_{20} = -0.25$, and (b) $\beta a_{10} = -\gamma a_{20} = 0.25$.

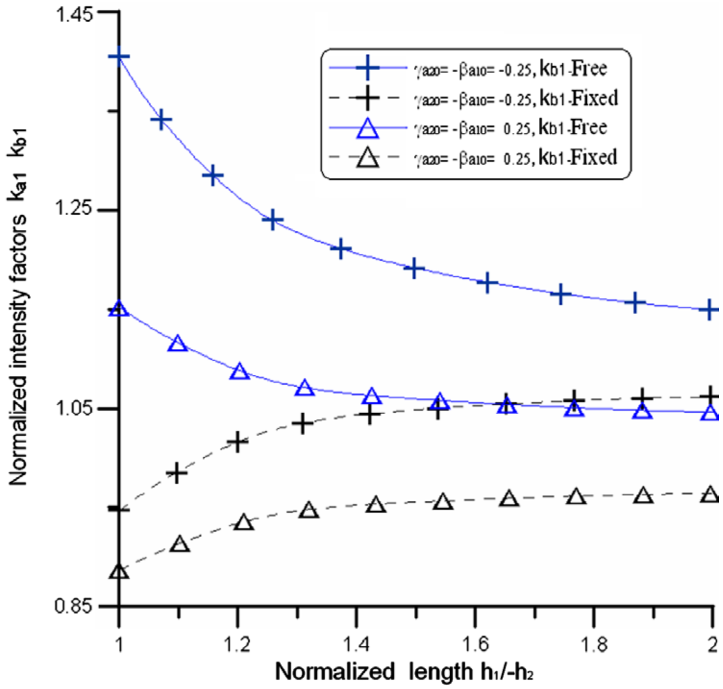

Fig. 4. Variations of normalized stress intensity factor k_{b1} with h_1 $-h_2$.

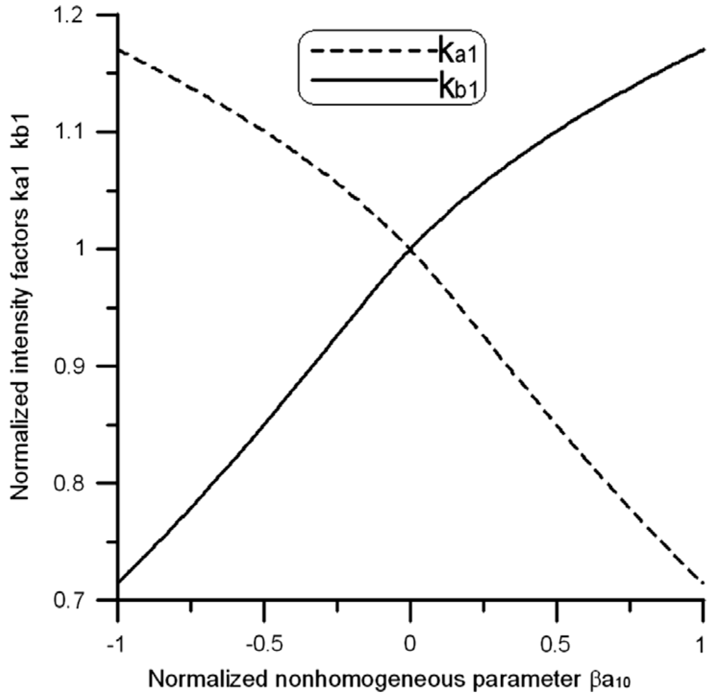

Fig. 5. Variations of normalized stress intensity factors with βa_{10} for a crack in an infinite medium.

In both case $\beta a_{10} = -\gamma a_{20} = -0.25$ and $\beta a_{10} = -\gamma a_{20} = 0.25$, the normalized stress intensity factor k_{b1} is higher when the boundary surface is traction-free than that when the boundary surface is fixed. In addition, the factor k_{b1} decreases when the traction-free boundary surface is moved to the right and vice versa for the fixed boundary surface case.

5.3. Numerical validation of degenerated problem 3: a crack in a FGM ($h_1 \to \infty$, $-h_2 \to \infty$, $2a_{20} = 0$, $\gamma = \beta$)

In this section, we will compare the numerical results of degenerated problem 3 with those of Erdogan (1985). Degenerated problem 3 is an infinite FGM containing a crack. Fig. 5 shows the results of the variations of normalized intensity factors with nonhomogeneous parameter βa_{10}. It agrees well with Fig. 2 in the paper Erdogan (1985).

6. Conclusions

The fracture behavior of two bonded cracked FGM strips has been studied. The effects of the nonhomogeneous material parameters, crack locations and edge boundary conditions on the stress intensity factors have been emphasized. From the results, it shows that (1) the stress intensity factors decrease with increase in the distance between two cracks; and (2) the stress intensity factors are larger when the crack tip is located in the stiffer elastic medium and is near the free boundary surface.

Appendix A. The kernels $k_i\,(x, t)$ ($i = 1, 2, \ldots, 18$) of Case 1 in Eq. (18) are:

$$k_1(x,t) = \frac{i}{2}\int_{-\infty}^{\infty} \frac{m_1}{\alpha} e^{i\alpha(t-x)} d\alpha \tag{A.1}$$

$$k_2(x,t) = e^{\frac{\beta}{2}(t-x)} \int_0^{\infty} \frac{2\alpha^2\alpha_2 p_1 e^{p_2h_1}[q_1q_2(e^{q_2h_2} - e^{q_1h_2}) + p_2(q_1e^{q_1h_2} - q_2e^{q_2h_2})]}{\Delta} e^{-\alpha_1(t+x)} d\alpha \tag{A.2}$$

$$k_3(x,t) = e^{\frac{\beta}{2}(t-x)} \int_0^{\infty} \frac{2\alpha^2\alpha_2 e^{p_1h_1}[q_1q_2(e^{q_1h_2} - e^{q_2h_2}) + p_2(q_2e^{q_2h_2} - q_1e^{q_1h_2})]}{\Delta} e^{\alpha_1(t-x)} d\alpha \tag{A.3}$$

$$k_4(x,t) = e^{\frac{\beta}{2}(t-x)} \int_0^{\infty} \frac{2\alpha^2\alpha_2 p_1 e^{p_1h_1}[(q_2e^{q_2h_2} - q_1e^{q_1h_2}) + p_1(e^{q_1h_2} - e^{q_2h_2})]}{\Delta} e^{-\alpha_1(t-x)} d\alpha \tag{A.4}$$

$$k_5(x,t) = e^{\frac{\beta}{2}(t-x)} \int_0^{\infty} \frac{2\alpha^2\alpha_2 p_1 e^{p_1h_1}[(q_1e^{q_1h_2} - q_2e^{q_2h_2}) + p_2(e^{q_2h_2} - e^{q_1h_2})]}{\Delta} e^{\alpha_1(t+x)} d\alpha \tag{A.5}$$

$$k_6(x,t) = e^{(\frac{\gamma}{2}t - \frac{\beta}{2}x)} \int_0^{\infty} \frac{2\alpha^2\alpha_1 p_2 e^{p_2h_1} e^{q_2h_2}(q_2 - q_1)}{\Delta} e^{(\alpha_2 t - \alpha_1 x)} d\alpha \tag{A.6}$$

$$k_7(x,t) = e^{(\frac{\gamma}{2}t - \frac{\beta}{2}x)} \int_0^{\infty} \frac{2\alpha^2\alpha_1 p_2 e^{p_2h_1} e^{q_1h_2}(q_1 - q_2)}{\Delta} e^{(-\alpha_2 t - \alpha_1 x)} d\alpha \tag{A.7}$$

$$k_8(x,t) = e^{(\frac{\gamma}{2}t - \frac{\beta}{2}x)} \int_0^{\infty} \frac{2\alpha^2\alpha_1 p_1 e^{p_1h_1} e^{q_2h_2}(q_1 - q_2)}{\Delta} e^{(\alpha_2 t + \alpha_1 x)} d\alpha \tag{A.8}$$

$$k_9(x,t) = e^{(\frac{\gamma}{2}t - \frac{\beta}{2}x)} \int_0^{\infty} \frac{2\alpha^2\alpha_1 p_1 e^{p_1h_1} e^{q_2h_2}(q_2 - q_1)}{\Delta} e^{(-\alpha_2 t + \alpha_1 x)} d\alpha \tag{A.9}$$

$$k_{10}(x,t) = \frac{i}{2}\int_{-\infty}^{\infty} \frac{m_2}{\alpha} e^{i\alpha(t-x)} d\alpha \tag{A.10}$$

$$k_{11}(x,t) = e^{\frac{\gamma}{2}(t-x)} \int_0^{\infty} \frac{2\alpha^2\alpha_1 e^{q_2h_2}[p_1p_2(e^{p_1h_1} - e^{p_2h_1}) + q_2(p_2e^{p_2h_1} - p_1e^{p_1h_1})]}{\Delta} e^{\alpha_2(t+x)} d\alpha \tag{A.11}$$

$$k_{12}(x,t) = e^{\frac{\gamma}{2}(t-x)} \int_0^{\infty} \frac{2\alpha^2\alpha_1 e^{q_1h_2}[p_1p_2(e^{p_2h_1} - e^{p_1h_1}) + q_2(p_1e^{p_1h_1} - p_2e^{p_2h_1})]}{\Delta} e^{-\alpha_2(t+x)} d\alpha \tag{A.12}$$

$$k_{13}(x,t) = e^{\frac{\gamma}{2}(t-x)} \int_0^{\infty} \frac{2\alpha^2\alpha_1 q_1 e^{q_1h_2}[q_1(e^{p_2h_1} - e^{p_1h_1}) + (p_1e^{p_1h_1} - p_2e^{p_2h_1})]}{\Delta} e^{\alpha_2(t+x)} d\alpha \tag{A.13}$$

$$k_{14}(x,t) = e^{\frac{\gamma}{2}(t-x)} \int_0^{\infty} \frac{2\alpha^2\alpha_1 e^{q_1h_2}[p_1p_2(e^{p_1h_1} - e^{p_2h_1}) + q_1(p_2e^{p_2h_1} - p_1e^{p_1h_1})]}{\Delta} e^{-\alpha_2(t+x)} d\alpha \tag{A.14}$$

$$k_{15}(x,t) = e^{(\frac{\beta}{2}t - \frac{\gamma}{2}x)} \int_0^{\infty} \frac{2\alpha^2\alpha_2 q_2 e^{p_2h_1} e^{q_2h_2}(p_1 - p_2)}{\Delta} e^{(-\alpha_1 t + \alpha_2 x)} d\alpha \tag{A.15}$$

$$k_{16}(x,t) = e^{(\frac{\beta}{2}t - \frac{\gamma}{2}x)} \int_0^{\infty} \frac{2\alpha^2\alpha_2 q_2 e^{p_1h_1} e^{q_2h_2}(p_2 - p_1)}{\Delta} e^{(\alpha_1 t + \alpha_2 x)} d\alpha \tag{A.16}$$

$$k_{17}(x,t) = e^{(\frac{\beta}{2}t - \frac{\gamma}{2}x)} \int_0^{\infty} \frac{2\alpha^2\alpha_2 q_1 e^{p_2h_1} e^{q_1h_2}(p_2 - p_1)}{\Delta} e^{(-\alpha_1 t - \alpha_2 x)} d\alpha \tag{A.17}$$

$$k_{18}(x,t) = e^{(\frac{\beta}{2}t - \frac{\gamma}{2}x)} \int_0^{\infty} \frac{2\alpha^2\alpha_2 q_1 e^{p_1h_1} e^{q_1h_2}(p_1 - p_2)}{\Delta} e^{(\alpha_1 t - \alpha_2 x)} d\alpha \tag{A.18}$$

where

$$\Delta = -2\alpha_1\alpha_2\alpha^2\left[(e^{p_1h_1} - e^{p_2h_1})(q_1e^{q_1h_2} - q_2e^{q_2h_2}) + (e^{q_2h_2} - e^{q_1h_2})(p_1e^{p_1h_1} - p_2e^{p_2h_1})\right]$$

Appendix B. The kernels $k_j\ (x, t)\ (j = 1, 2, \ldots, 18)$ of Case 2 are:

$$k_1(x,t) = \frac{i}{2}\int_{-\infty}^{\infty}\frac{m_1}{\alpha}e^{i\alpha(t-x)}d\alpha \tag{B.1}$$

$$k_2(x,t) = e^{\frac{\beta}{2}(t-x)}\int_0^{\infty}\frac{2\alpha^2\alpha_2p_1q_1q_2e^{p_2h_1}[p_2(e^{q_2h_2} - e^{q_1h_2}) + (q_2e^{q_1h_2} - q_1e^{q_2h_2})]}{\Delta_1}e^{-\alpha_1(t+x)}d\alpha \tag{B.2}$$

$$k_3(x,t) = e^{\frac{\beta}{2}(t-x)}\int_0^{\infty}\frac{2\alpha^2\alpha_2p_2q_1q_2e^{p_1h_1}[p_2(e^{q_1h_2} - e^{q_2h_2}) + (q_1e^{q_2h_2} - q_2e^{q_1h_2})]}{\Delta_1}e^{\alpha_1(t-x)}d\alpha \tag{B.3}$$

$$k_4(x,t) = e^{\frac{\beta}{2}(t-x)}\int_0^{\infty}\frac{2\alpha^2\alpha_2p_1q_1q_2e^{p_1h_1}[p_2(e^{q_1h_2} - e^{q_2h_2}) + (q_1e^{q_2h_2} - q_2e^{q_1h_2})]}{\Delta_1}e^{-\alpha_1(t-x)}d\alpha \tag{B.4}$$

$$k_5(x,t) = e^{\frac{\beta}{2}(t-x)}\int_0^{\infty}\frac{2\alpha^2\alpha_2p_2q_1q_2e^{p_1h_1}[p_1(e^{q_2h_2} - e^{q_1h_2}) + (q_2e^{q_1h_2} - q_1e^{q_2h_2})]}{\Delta_1}e^{\alpha_1(t+x)}d\alpha \tag{B.5}$$

$$k_6(x,t) = e^{(\frac{\gamma}{2}t-\frac{\beta}{2}x)}\int_0^{\infty}\frac{2\alpha^2\alpha_1p_1p_2q_1(q_1 - q_2)e^{p_2h_1}e^{q_2h_2}}{\Delta_1}e^{(\alpha_2t-\alpha_1x)}d\alpha \tag{B.6}$$

$$k_7(x,t) = e^{(\frac{\gamma}{2}t-\frac{\beta}{2}x)}\int_0^{\infty}\frac{2\alpha^2\alpha_1p_1p_2q_2(q_2 - q_1)e^{p_2h_1}e^{q_1h_2}}{\Delta_1}e^{(-\alpha_2t-\alpha_1x)}d\alpha \tag{B.7}$$

$$k_8(x,t) = e^{(\frac{\gamma}{2}t-\frac{\beta}{2}x)}\int_0^{\infty}\frac{2\alpha^2\alpha_1p_1p_2q_1(q_2 - q_1)e^{p_1h_1}e^{q_2h_2}}{\Delta_1}e^{(\alpha_2t+\alpha_1x)}d\alpha \tag{B.8}$$

$$k_9(x,t) = e^{(\frac{\gamma}{2}t-\frac{\beta}{2}x)}\int_0^{\infty}\frac{2\alpha^2\alpha_1p_1p_2q_2(q_1 - q_2)e^{p_1h_1}e^{q_1h_2}}{\Delta_1}e^{(-\alpha_2t+\alpha_1x)}d\alpha \tag{B.9}$$

$$k_{10}(x,t) = \int_{-\infty}^{\infty}\frac{im_2}{2\alpha}e^{i\alpha(t-x)}d\alpha \tag{B.10}$$

$$k_{11}(x,t) = e^{\frac{\gamma}{2}(t-x)}\int_0^{\infty}\frac{2\alpha^2\alpha_1p_1p_2q_1e^{q_2h_2}[q_2(e^{p_1h_1} - e^{p_2h_1}) + (p_1e^{p_2h_1} - p_2e^{p_1h_1})]}{\Delta_1}e^{\alpha_2(t+x)}d\alpha \tag{B.11}$$

$$k_{12}(x,t) = e^{\frac{\gamma}{2}(t-x)}\int_0^{\infty}\frac{2\alpha^2\alpha_1p_1p_2q_2e^{q_1h_2}[q_2(e^{p_2h_1} - e^{p_1h_1}) + (p_1e^{p_1h_1} - p_2e^{p_2h_1})]}{\Delta_1}e^{-\alpha_2(t-x)}d\alpha \tag{B.12}$$

$$k_{13}(x,t) = e^{\frac{\gamma}{2}(t-x)}\int_0^{\infty}\frac{2\alpha^2\alpha_1p_1p_2q_1e^{q_1h_2}[q_2(e^{p_2h_1} - e^{p_1h_1}) + (p_2e^{p_1h_1} - p_1e^{p_2h_1})]}{\Delta_1}e^{\alpha_2(t-x)}d\alpha \tag{B.13}$$

$$k_{14}(x,t) = e^{\frac{\gamma}{2}(t-x)}\int_0^{\infty}\frac{2\alpha^2\alpha_1p_1p_2q_2e^{q_1h_2}[q_1(e^{p_1h_1} - e^{p_2h_1}) + (p_1e^{p_2h_1} - p_2e^{p_1h_1})]}{\Delta_1}e^{-\alpha_2(t+x)}d\alpha \tag{B.14}$$

$$k_{15}(x,t) = e^{(\frac{\beta}{2}t-\frac{\gamma}{2}x)}\int_0^{\infty}\frac{2\alpha^2\alpha_2p_1q_1q_2(p_2 - p_1)e^{p_2h_1}e^{q_2h_2}}{\Delta_1}e^{(-\alpha_1t+\alpha_2x)}d\alpha \tag{B.15}$$

$$k_{16}(x,t) = e^{(\frac{\beta}{2}t-\frac{\gamma}{2}x)}\int_0^{\infty}\frac{2\alpha^2\alpha_2p_2q_1q_2(p_1 - p_2)e^{p_1h_1}e^{q_2h_2}}{\Delta_1}e^{(\alpha_1t+\alpha_2x)}d\alpha \tag{B.16}$$

$$k_{17}(x,t) = e^{(\frac{\beta}{2}t-\frac{\gamma}{2}x)}\int_0^{\infty}\frac{2\alpha^2\alpha_2p_1q_1q_2(p_1 - p_2)e^{p_2h_1}e^{q_1h_2}}{\Delta_1}e^{(-\alpha_1t-\alpha_2x)}d\alpha \tag{B.17}$$

$$k_{18}(x,t) = e^{(\frac{\beta}{2}t-\frac{\gamma}{2}x)}\int_0^{\infty}\frac{2\alpha^2\alpha_2p_2q_1q_2(p_2 - p_1)e^{p_1h_1}e^{q_1h_2}}{\Delta_1}e^{(\alpha_1t-\alpha_2x)}d\alpha \tag{B.18}$$

where

$$\Delta_1 = 2p_1p_2q_1q_2\alpha_1\alpha_2\left[(e^{q_1h_2} - e^{q_2h_2})(p_1e^{p_2h_1} - p_2e^{p_1h_1}) + (e^{p_1h_1} - e^{p_2h_1})(q_2e^{q_1h_2} - q_1e^{q_2h_2})\right]$$

Appendix C. Degenerated the edge boundary h_2 to minus infinity

$$k_1(x,t) = \frac{i}{2}\int_{-\infty}^{\infty}\frac{m_1}{\alpha}e^{i\alpha(t-x)}d\alpha$$

$$k_2(x,t) = e^{\frac{\beta}{2}(t-x)}\int_0^{\infty}\frac{q_2(p_2 - q_1)e^{\alpha_1h_1}}{\alpha_1[(p_1 - q_2)e^{-\alpha_1h_1} + (q_2 - p_2)e^{\alpha_1h_1}]}e^{-\alpha_1(t+x)}d\alpha \tag{C.1}$$

$$k_3(x,t) = e^{\frac{\beta}{2}(t-x)} \int_0^\infty \frac{q_2(q_1-p_2)e^{-\alpha_1 h_1}}{\alpha_1[(p_1-q_2)e^{-\alpha_1 h_1} + (q_2-p_2)e^{\alpha_1 h_1}]} e^{\alpha_1(t-x)} d\alpha \tag{C.2}$$

$$k_4(x,t) = e^{\frac{\beta}{2}(t-x)} \int_0^\infty \frac{p_1(p_1-q_2)e^{-\alpha_1 h_1}}{\alpha_1[(p_1-q_2)e^{-\alpha_1 h_1} + (q_2-p_2)e^{\alpha_1 h_1}]} e^{-\alpha_1(t-x)} d\alpha \tag{C.3}$$

$$k_5(x,t) = e^{\frac{\beta}{2}(t-x)} \int_0^\infty \frac{p_1(p_2-q_2)e^{-\alpha_1 h_1}}{\alpha_1[(p_1-q_2)e^{-\alpha_1 h_1} + (q_2-p_2)e^{\alpha_1 h_1}]} e^{\alpha_1(t+x)} d\alpha \tag{C.4}$$

$$k_6(x,t) = e^{(\frac{\gamma}{2}t-\frac{\beta}{2}x)} \int_0^\infty \frac{p_2(q_1-q_2)e^{\alpha_1 h_1}}{\alpha_2[(p_1-q_2)e^{-\alpha_1 h_1} + (q_2-p_2)e^{\alpha_1 h_1}]} e^{(\alpha_2 t-\alpha_1 x)} d\alpha \tag{C.5}$$

$$k_7(x,t) = e^{(\frac{\gamma}{2}t-\frac{\beta}{2}x)} \int_0^\infty \frac{p_1(q_2-q_1)e^{-\alpha_1 h_1}}{\alpha_2[(p_1-q_2)e^{-\alpha_1 h_1} + (q_2-p_2)e^{\alpha_1 h_1}]} e^{(\alpha_2 t+\alpha_1 x)} d\alpha \tag{C.6}$$

$$k_{10}(x,t) = \frac{i}{2} \int_{-\infty}^\infty \frac{m_2}{\alpha} e^{i\alpha(t-x)} d\alpha \tag{C.7}$$

$$k_{11}(x,t) = e^{\frac{\gamma}{2}(t-x)} \int_0^\infty \frac{e^{\alpha_1 h_1}\{p_1 p_2(e^{\alpha_1 h_1} - e^{-\alpha_1 h_1}) - q_2(p_2 e^{\alpha_1 h_1} - p_1 e^{-\alpha_1 h_1})\}}{\alpha_2[(p_1-q_2)e^{-\alpha_1 h_1} + (q_2-p_2)e^{\alpha_1 h_1}]} e^{\alpha_2(t+x)} d\alpha \tag{C.8}$$

$$k_{15}(x,t) = e^{(\frac{\beta}{2}t-\frac{\gamma}{2}x)} \int_0^\infty \frac{q_2(p_2-p_1)e^{\alpha_1 h_1}}{\alpha_1[(p_1-q_2)e^{-\alpha_1 h_1} + (q_2-p_2)e^{\alpha_1 h_1}]} e^{(-\alpha_1 t+\alpha_2 x)} d\alpha \tag{C.9}$$

$$k_{16}(x,t) = e^{(\frac{\beta}{2}t-\frac{\gamma}{2}x)} \int_0^\infty \frac{q_2(p_1-p_2)e^{-\alpha_1 h_1}}{\alpha_1[(p_1-q_2)e^{-\alpha_1 h_1} + (q_2-p_2)e^{\alpha_1 h_1}]} e^{(\alpha_1 t+\alpha_2 x)} d\alpha \tag{C.10}$$

Appendix D. Eq. (19) can be solved after reducing them into the following Chebyshev polynomial equations:

$$\left\{\begin{array}{l}
\tau_1(x_r) = \mu_0 e^{\beta(a_{10}x_r+c)} \left\{\begin{array}{l} \sum_{k=1}^n \frac{1}{n} F_1(t_k)\left[\frac{1}{t_k-x_r} + \pi(h_1(x_r,t_k) + k_2(x_r,t_k) + k_4(x_r,t_k) + k_4(x_r,t_k) + k_5(x_r,t_k))\right] \\ + \sum_{k=1}^n \frac{\pi}{n} F_2(t_k)[k_6(x_r,t_k) + k_7(x_r,t_k) + k_8(x_r,t_k) + k_9(x_r,t_k)] \end{array}\right\} \\
\tau_2(x_r) = \mu_0 e^{\gamma(a_{20}x_r+c)} \left\{\begin{array}{l} \sum_{k=1}^n \frac{1}{n} F_2(t_k)\left[\frac{1}{t_k-x_r} + \pi(h_{10}(x_r,t_k) + k_{11}(x_r,t_k) + k_{12}(x_r,t_k) + k_{13}(x_r,t_k) + k_{14}(x_r,t_k))\right] \\ + \sum_{k=1}^n \frac{\pi}{n} F_1(t_k)[k_{15}(x_r,t_k) + k_{16}(x_r,t_k) + k_{17}(x_r,t_k) + k_{18}(x_r,t_k)] \end{array}\right\} \\
\sum_{k=1}^n \frac{\pi}{n} F_1(t_k) = 0, \sum_{k=1}^n \frac{\pi}{n} F_2(t_k) = 0
\end{array}\right\} \tag{D1}$$

where $t_k = \cos(2k\text{-}1)\pi/2n$, $(k = 1, 2, \ldots, n)$; $x_r = \cos(r\pi/n)$, $(r = 1, 2, \ldots, n-1)$ are the nodes satisfy Chebyshev polynomial of the first and second kind respectively, Eq. (D1) involve $2n$ simultaneous linear equations to solve $2n$ unknowns $F_i(t_k)$ (i = 1, 2 and k = 1, 2, ..., n).

References

Choi, H.J., 1996. Bonded dissimilar strips with a crack perpendicular to the functionally graded interface. International Journal of Solid and Structures 33, 4101–4117.

Erdogan, F., 1985. The crack problem for bonded nonhomogeneous materials under antiplane shear loading. Transactions of the ASME, Journal of Applied Mechanics 52, 823–828.

Erdogan, F., Gupta, G.D., Cook, T.S., 1973. Numerical solution of singular integral equations. In: Sih, G.C. (Ed.), Mechanics of Fracture. 1. Method of Analysis and Solution of Crack Problem. Noordhoff International Publishing, Leyden, The Netherlands.

Erdogan, F., Kaya, A.C., Joseph, P.F., 1991. The mode III crack problem in bonded materials with a nonhomogeneous interfacial zone. Transactions of the ASME, Journal of Applied Mechanics 58, 419–427. Chapter 7.

Erdogan, F., Wu, B.H., 1997. The surface crack problem for a plate with functionally graded properties. Transactions of the ASME, Journal of Applied Mechanics 64, 449–456.

Fotuhi, A.R., Fariborz, S.J., 2006. Anti-plane analysis of a functionally graded strip with multiple cracks. International Journal of Solids and Structures 43, 1239–1252.

Gao, L.C., Wu, L.Z., Ma, T.Z., 2004. Mode I crack for a functionally graded orthotropic strip. European Journal of Mechanics 23, 219–234.

Jin, Z.H., Paulino, G.H., 2002. A viscoelastic functionally graded strip containing a crack subjected to in-plane loading. Engineering Fracture Mechanics 69, 1769–1790.

Jin, B., Soh, A.K., Zhong, Z., 2003. Propagation of an anti-plane moving crack in a functionally graded piezoelectric strip. Archive of Applied Mechanics 73, 252–260.

Long, X., Delale, F., 2004. The general problem for an arbitrarily orient crack in a FGM layer. International Journal of Fracture 129, 221–238.

Muskhhelishvili, N.I., 1953. Singular Integral Equations. Noordhoff International Publishing, Groningen, The Netherlands.

Noda, N., Jin, Z.H., 1993. Thermal stress intensity factors for a crack in a strip of a functionally gradient material. International Journal of Solids and Structures 30, 1039–1056.

Ueda, S., 2005. Impact response of a functionally graded piezoelectric plate with a vertical crack. Theoretical and Applied Mechanics 44, 329–342.

Ueda, S., Mukai, T., 2002. The surface crack problem for a layered elastic medium with a functionally graded nonhomogeneous interface. JSME International Journal, Series A 45, 371–378.

Wang, B.L., Mai, Y.W., Sun, Y.G., 2003. Anti-plane fracture of a functionally graded material strip. European Journal of Mechanics A/Solids 22, 357–368.